

Monodromy, Chern Classes, and their Physical Significance

Research Thesis

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1. INTRODUCTION

Given an n -dimensional mechanical system, with n -constants of motion, action-angle coordinates are a tool for solving for the equations of motion. In a rough sense, we pick out periodic trajectories in the configuration space, coordinatize these by their corresponding periods, and “fill out” our coordinate system with actions associated to the angles.

The purpose of this thesis is to explore the mathematics behind these action-angle coordinates, and examine what happens in the “global” case. We begin with the analytic formulation, within which we follow the historical and physical development of actions and angles. We then turn to the underlying topology, and examine actions and angle in this context. In particular, we will examine the two main topological restrictions that exist when attempting to construct global action-angle coordinates: Chern classes and monodromy. This is followed by a discussion of the spherical pendulum. This system is interesting because it is a nice enough mechanical system for which our topological obstructions do in fact keep us from constructing global action angle coordinates. Finally, we give a taste of some of the more modern accounts of this theory. In particular we briefly touch on Lie algebra actions on symplectic manifolds, and relate them to the other discussions in the thesis.

2. ANALYTIC CONSIDERATIONS

In this section we follow the analytic construction of action-angle coordinates. In particular, we develop the physical intuition and formulations necessary to understand the tools, followed by some more mathematical considerations.

2.1. Fundamental Concepts. In all that follows, we will concern ourselves with some mechanical system. For simplicity, we will just consider systems comprised of a single particle, though much of the theory readily generalizes to other types of systems.

The *configuration space* of a system consists of all possible configurations the particle can take. In the case of the simple pendulum, the configuration space is just a circle S^1 ; for spherical pendulums, the configuration space is S^2 ; for a particle moving with no external force acting on it, the configuration space is a straight line; etc. If

we give the location of our particle within the system by coordinates $q = (q_1, \dots, q_n)$, then the configuration space is the space spanned by all these possible q . Also, since we are analyzing the motion of our particle through time, q is actually a function of time $q = q(t)$. '

We will often have cause to assign to our system certain quantities. The foremost of these are the *kinetic energy* KE and *potential energy* PE . The kinetic energy is given by

$$KE = \frac{1}{2}m\langle\dot{q}, \dot{q}\rangle,$$

where m is the mass of the particle, and \langle, \rangle is the (Riemannian) metric of the configuration space. In all of the situations we consider, we will utilize the usual Euclidean metric. The potential energy is determined completely by the system. For mechanical systems, the potential energy arises from gravity, springs, or any other force that pushes on the particle.

As an example, we consider the spherical pendulum. Indeed, this example will play an important role in later discussions. The configuration space of this system is S^2 , the sphere, so we can use polar coordinates

$$\begin{aligned} x &= \sin \phi \cos \theta, \\ y &= \sin \phi \sin \theta, \\ z &= \cos \phi, \end{aligned}$$

with $\phi \in [0, \pi]$ and $\theta \in [0, 2\pi)$. Here, ϕ corresponds to the longitude, whereas θ corresponds to the latitude, and we will ignore the singular issues that arise when $\phi = 0, \pi$ in terms of our coordinate chart. Note that both ϕ and θ are functions of time, and that $\phi = q_1$ and $\theta = q_2$ are our coordinate system, not x, y, z . On a conceptual level, the spherical pendulum is a 2-dimensional system, so it wouldn't make sense to describe a particle's location using three coordinates.

To simplify computations, we assume the particle has unit mass and gravity is scaled to have unit magnitude. Then the potential energy is given by

$$PE = mgq_3 = \cos \phi,$$

and the kinetic energy is

$$\begin{aligned} KE &= \frac{1}{2}m\langle\dot{q}, \dot{q}\rangle \\ &= \frac{(\dot{q}_1, \dot{q}_2, \dot{q}_3) \cdot (\dot{q}_1, \dot{q}_2, \dot{q}_3)}{2} \\ &= \frac{\dot{\theta}^2 \sin^2 \phi + \dot{\phi}^2}{2}, \end{aligned}$$

which can be verified by direct calculation.

2.2. Lagrangian Mechanics. Lagrangian mechanics is a reformulation of Newtonian mechanics, in which we consider paths through a system's configuration space M which minimize a certain functional. The novelty in this formulation is that more powerful analytic tools can be brought in to help solve for the equations of motion, or at least let us glean some information about the behaviour of the particle.

The tools of interest come from the calculus of variations, since we are interested in minimizing the quantity

$$\int_{q(0)}^{q(1)} \mathcal{L}(q, \dot{q}, t) dt.$$

The term $\mathcal{L}(q, \dot{q}, t)$ is called the *Lagrangian* of the system, and its integral along the path $q: [0, 1] \rightarrow M$ is called the *action* of the system. The Principle of Least Action stipulates that a particle will follow a path of least action, so to know the path of the particle comes down to finding such minimal paths. The calculus of variations then tells us how to find such a path: solve the differential equations

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \frac{\partial \mathcal{L}}{\partial q_i} = 0, \quad 1 \leq i \leq n.$$

These are known as the Euler-Lagrange equations, and do indeed supply minimal solutions for the action of a system.

For all cases we will be concerned with, we define the Lagrangian of our system as

$$\mathcal{L} = KE - PE.$$

Turning back to the example of S^2 , we find that

$$\mathcal{L}(q, \dot{q}, t) = \frac{\dot{\theta}^2 \sin^2 \phi + \dot{\phi}^2}{2} - \cos \phi.$$

Utilizing the Euler-Lagrange equations gives us

$$\begin{aligned}
 0 &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_1} - \frac{\partial \mathcal{L}}{\partial q_1}, \\
 &= \frac{d}{dt} (\dot{\phi}) - \dot{\theta}^2 \sin \phi (1 + \cos \phi), \\
 &= \ddot{\phi} - \dot{\theta}^2 \sin \phi (1 + \cos \phi),
 \end{aligned}$$

and

$$\begin{aligned}
 0 &= \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_2} - \frac{\partial \mathcal{L}}{\partial q_2}, \\
 &= \frac{d}{dt} (\dot{\theta} \sin^2 \phi) \\
 &= \ddot{\theta} \sin^2 \phi + 2\dot{\theta} \dot{\phi} \sin \phi \cos \phi.
 \end{aligned}$$

Notice that these two (second-order) differential equations don't admit solutions by inspection, and in fact are rather difficult to obtain explicit solutions for. This is characteristic of much of classical mechanics though; the equations become highly nonlinear and, in this case elliptic integrals pop up, and so we turn to analyzing the qualitative behaviour of the system.

2.3. Hamiltonian Mechanics. Hamiltonian mechanics is the proper reformulation from Lagrangian mechanics if we hope to understand the qualitative structure of the system. In brief, we make a coordinate transformation from the q 's and \dot{q} 's to q 's and p 's, where each p is a conjugate momentum to \dot{q} . This transformation is made via the Legendre transform, and turns our problem of solving n -second order differential equations into solving $2n$ -first order differential equations. Here, however, we will proceed at a slightly less than formal level, sidestepping discussions of the Legendre transform and jumping right to the conclusion.

As before, we consider a particle with coordinates q_i , velocities \dot{q}_i , and Lagrangian \mathcal{L} . Define the *conjugate momenta* to be

$$p_i := \frac{\partial \mathcal{L}}{\partial \dot{q}_i},$$

and the system's *Hamiltonian* \mathcal{H} to be

$$\mathcal{H}(q_i, p_i, t) := \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t).$$

Now take the total differential of \mathcal{H} . On the left hand side, we get

$$d\mathcal{H} = \sum_i \left(\frac{\partial \mathcal{H}}{\partial q_i} dq_i + \frac{\partial \mathcal{H}}{\partial p_i} dp_i + \frac{\partial \mathcal{H}}{\partial t} dt \right). \quad (1)$$

On the right hand side,

$$d\mathcal{H} = \sum_i \left(\dot{q}_i dp_i + p_i d\dot{q}_i - \frac{\partial \mathcal{L}}{\partial q_i} dq_i - \frac{\partial \mathcal{L}}{\partial \dot{q}_i} d\dot{q}_i \right) - \frac{\partial \mathcal{L}}{\partial t} dt. \quad (2)$$

Now by the Euler-Lagrange equations, we have

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i},$$

so that

$$\dot{p}_i = \frac{\partial \mathcal{L}}{\partial q_i}.$$

Recalling as well that $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$, equation (2) becomes

$$d\mathcal{H} = \sum_i (\dot{q}_i dp_i + p_i d\dot{q}_i - \dot{p}_i dq_i - p_i d\dot{q}_i) - \frac{\partial \mathcal{L}}{\partial t} dt. \quad (3)$$

Subtracting (3) from (1), and collecting like terms, we get

$$0 = \sum_i \left[\left(\frac{\partial \mathcal{H}}{\partial q_i} + \dot{p}_i \right) dq_i + \left(\frac{\partial \mathcal{H}}{\partial p_i} - \dot{q}_i \right) dp_i + \left(\frac{\partial \mathcal{H}}{\partial t} + \frac{\partial \mathcal{L}}{\partial t} \right) dt \right].$$

From this last equation, we derive *Hamilton's equations*

$$\begin{aligned} \dot{q}_i &= \frac{\partial \mathcal{H}}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial \mathcal{H}}{\partial q_i}, \end{aligned}$$

in addition to

$$\frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}.$$

Plugging Hamilton's equations back into the total differential of \mathcal{H} shows that

$$\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t},$$

so the Hamiltonian is a constant if the Lagrangian does not depend on time. Lucky for us, our Lagrangians will be time independent.

Turning again back to our example of the spherical pendulum, we have

$$\begin{aligned} p_1 &= \frac{\partial \mathcal{L}}{\partial \dot{q}_1} = \dot{\phi}, \\ p_2 &= \frac{\partial \mathcal{L}}{\partial \dot{q}_2} = \dot{\theta} \sin^2(\phi). \end{aligned}$$

Equivalently, we can write

$$\begin{aligned} \dot{q}_1 &= p_1, \\ \dot{q}_2 &= \frac{p_2}{\sin^2 \phi}. \end{aligned}$$

The corresponding Hamiltonian is then

$$\begin{aligned} \mathcal{H}(q_i, p_i, t) &= \sum_i p_i \dot{q}_i - \mathcal{L}(q_i, \dot{q}_i, t) \\ &= (p_1)^2 + \frac{(p_2)^2}{\sin^2 \phi} - \frac{(p_2)^2}{2 \sin^2 \phi} - \frac{(p_1)^2}{2} + \cos \phi \\ &= \frac{(p_1)^2}{2} + \frac{(p_2)^2}{2 \sin^2 \phi} + \cos \theta. \end{aligned}$$

2.4. Hamilton-Jacobi Theory. Since the Hamiltonian formulation takes us from n second order differential equations to $2n$ first order equations, one could naturally wonder if it is possible to make any further reductions or simplifications. One such way is through Hamilton-Jacobi theory.

The main idea is to perform a “coordinate transformation” $F = F(q, P, t)$ such that one Hamiltonian $H = H(q, p, t)$ can be related to another $K = K(Q, P, t)$ by the equation

$$H = K + \frac{\partial F}{\partial t}.$$

The function F isn’t, strictly speaking, a map from (q, P) to (Q, p) , as one might think a coordinate transformation would do. It’s usefulness becomes apparent in that it gives us a way of relating our two coordinate systems. Moreover, we have the relations

$$p = -\frac{\partial F}{\partial q} \tag{4}$$

$$Q = -\frac{\partial F}{\partial P}. \tag{5}$$

To see this, express the Lagrangian in terms of both H and $K + \frac{dF}{dt}$, i.e.

$$\begin{aligned}\mathcal{L}(q, \dot{q}, t) &= \dot{q}p - H(q, p, t) \\ \mathcal{L}(Q, \dot{Q}, t) &= \dot{Q}P - K(Q, P, t) - \frac{dF}{dt}.\end{aligned}$$

Note that the total time derivative of F does not affect the value of the action, since we assumed that any variation of the action had fixed endpoints. Now since each Hamiltonian represents the same system, we have $\mathcal{L}(q, p, t) = \mathcal{L}(Q, P, t)$. Taking the total differential of each side, and equating like terms, gives us the desired equations (4) and (5).

Since the choice of K is up to our discretion, we usually choose K to be a constant (particularly $K = 0$). Thus, we are led to the equation

$$H(q, p, t) + \frac{\partial F}{\partial t} = 0,$$

which we can reexpress as

$$H(q, \frac{\partial F}{\partial q}, t) + \frac{\partial F}{\partial t} = 0.$$

This is a single first order partial differential equation of F , so in many regards much simpler than our previous $2n$ first order equations.

Also, since K was assumed to be constant, we have that Q and P are also constants (from Hamilton's equations of motion). Writing $Q = \alpha$ and $P = \beta$, we have

$$F = F(q, \alpha, t),$$

so

$$\beta = \frac{\partial F}{\partial \alpha} = \frac{\partial F(q, \alpha, t)}{\partial \alpha}.$$

If this last equation can be inverted to solve for q , then

$$q = q(\alpha, \beta, t),$$

and subsequently

$$p = \frac{\partial F(q(\alpha, \beta, t), \alpha, t)}{\partial q},$$

so that

$$p = p(\alpha, \beta, t).$$

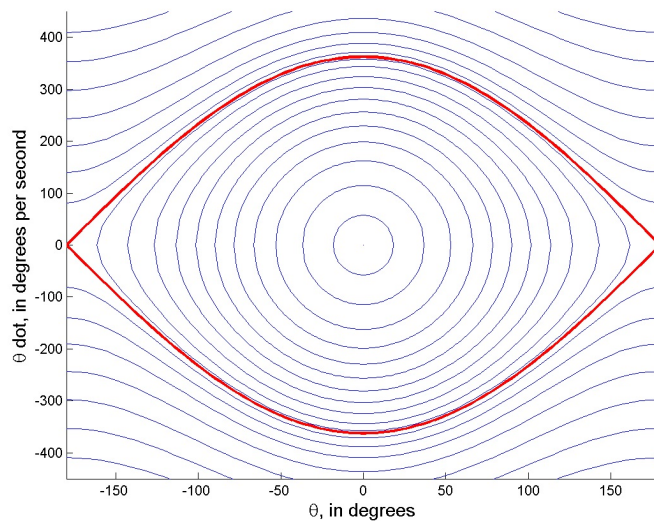
Fixing constants α, β , which are determined by the system's initial conditions, we have the recognizable form $q = q(t), p = p(t)$, i.e. we have a solution for the system's equations of motion.

In general, however, we are not so interested in the exact form of F so much as its partial derivatives.

2.5. Action-Angle Coordinates; the Physical Way. We now restrict ourselves to the case of systems with periodic motion. In particular, we are interested in finding suitable coordinate systems for the system which highlight the periods. In [Go], Goldstein makes a distinction between two types of periodic motion:

- *libration*, in which the coordinates p and q both have the same frequency. In other words, the trajectory of a particle traces a closed loop in the system's phase space.
- *rotation*, in which the phase space trajectory is invariant under translations.

Both types of periodic motion are best illustrated by means of the simple pendulum, whose phase diagram is included below:



Motions of the libration type correspond to level sets of small total energy, while motions of the rotation type correspond to level sets of high total energy.

We define the *action variables* by

$$J = \oint pdq.$$

If the motion is of libration type, Stokes' theorem tells us J is the area enclosed by the closed loop determined by q . If the motion is of rotation type, then the integral is taken over a complete period of rotation.

Recall that, under a suitable canonical transformation, we have

$$p = \frac{\partial F(q, \alpha)}{\partial q}.$$

Thus, we can also express action as

$$J = \oint pdq = \oint \frac{\partial F(q, \alpha)}{\partial q} dq,$$

so J is determined completely by α . We now use the J as our new conjugate momenta, so $H = H(q, J, t)$.

3. TOPOLOGICAL CONSIDERATIONS

In this section we develop the geometric and topological tools necessary to answer the question: “When can we construct global action-angle coordinates?” After some preliminary discussions, we examine the geometry behind the local constructions exemplified by Arnold's invariant tori theorem. Following this, we explore the global question, particularly the topological obstructions to global action-angle coordinates. One nice example of a system that does not admit global coordinates is the spherical pendulum, which we briefly discuss. We also introduce the momentum map for this system, which will give a nice segue into some more modern developments in this field of mathematics.

3.1. Hamiltonian Vector Fields and Poisson Brackets. The departure from the analytic constructions given in the last section are through the use of differential forms. Recall that we coordinatized the phase (cotangent) space of our system via the generalized coordinates q^i and the conjugate momenta p_i . We then define the *canonical 1-form* λ to be the form

$$\lambda = \sum_i p_i dq^i.$$

These are also the forms which appeared in the definition of our action coordinates, where the integrals were taken over different periods of the system. Another important form is the *canonical area form*, defined by

$$\omega = d\lambda = \sum_i dp_i \wedge dq^i.$$

It turns out that ω is non-degenerate, and so the cotangent bundle is a symplectic manifold. Taking the inner product of ω with a tangent vector v , denoted $i_v\omega$, gives us a 1-form, so ω gives a bijection between the tangent and cotangent bundles of our manifold.

If $H: M \rightarrow \mathbb{R}$ is any map from our $2n$ -dimensional symplectic manifold M , where again ω is the canonical area form on T^*M , a special role is played by vector fields X_H that satisfy

$$i_{X_H}\omega = -dH.$$

We call X_H the *Hamiltonian vector field* associated with the Hamiltonian H . In the physical situations above, M is actually the cotangent space to the configuration space N of our system. It is important to keep in mind where our functions are defined and what kind of spaces we are playing with; since the original Hamiltonians were functions of both position and momentum, we'd expect an arbitrary Hamiltonian-type function to be defined on a $2n$ -dimensional symplectic space.

We will also have cause to refer to the *Poisson bracket* of two Hamiltonians H and K , defined by

$$\{H, K\} := dK(X_H) = \omega(X_H, X_K).$$

It is quick to verify the following three properties of this bracket:

$$\begin{aligned} \{H, K\} &= -\{K, H\}, \\ [X_H, X_K] &= X_{\{H, K\}}, \\ \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} &= 0, \end{aligned}$$

where f, g, h are three Hamiltonians.

3.2. Arnold's Invariant Tori. The first major result for action-angle coordinates is a sort of local structure theorem due to Arnold, and independently to Markus and Meyer. The setting is as follows.

For an n -dimensional mechanical system we consider the cotangent bundle M , which is of course a $2n$ -dimensional symplectic manifold. We then call a system of k functions $\{f_i\}$ defined on M *in involution* if each of their pairwise Poisson brackets vanish, i.e. $\{f_i, f_j\} = 0$. Let

$M_r = \{x \in M \mid df_i(x) \text{ are linearly independent}\}$, so that M_r is the set of regular values of the function $f = (f_1, \dots, f_k)$. Finally, let $M_{r,c}$ be the subset of M_r consisting of compact fibers: $M_{r,c} = \{f^{-1}(x) \text{ compact} \mid x \in \mathbb{R}^n\}$.

Theorem 3.1. *$M_{r,c}$ is an open subset of M , and for each $x \in M_{r,c}$, there is an open neighborhood U of the fiber $f^{-1}(x)$ such that we have a diffeomorphism $(a, \alpha): U \mapsto V \times (\mathbb{R}/\mathbb{Z})^n$ with V open in \mathbb{R}^n , such that $a = \xi \cdot f$ for some diffeomorphism $\xi: f(U) \rightarrow V$ and*

$$\omega = \sum_i d\alpha_i \wedge da_i.$$

The key idea is that, restricting to compact fibers, we can carry out the action-angle coordinate construction. In [D1] one can find the two proofs by Markus and Meyer, followed by the proof by Arnold. In fact Arnold's proof gives an explicit construction of these coordinates, and can be found in many other modern treatments of classical mechanics (see for example, [S]).

The main step in Arnold's proof is constructing an action of \mathbb{R}^n on the Hamiltonian flows X_{f_i} of a point in the manifold M . This Lie group action is then associated to each of the compact fibers, and, by compactness, we conclude that the fibers must be tori. It is from these tori that we pick out our angles, and by integrating the Liouville form along generators for the first homology give us our corresponding actions.

3.3. Global Considerations. The global case is given as a statement about a $2n$ -dimensional symplectic manifold (M, ω) with a fibration $\pi: M \rightarrow B$, where B is an n -dimensional manifold and each fiber $\pi^{-1}(x), x \in B$, is a compact, connected Lagrange submanifold of M . A Lagrange submanifold F is a submanifold of M such that ω vanishes identically on F . Turning back to local theorem above, the assumption that each fiber is a Lagrange submanifold is meant to generalize the property of our integrals of motion being in involution.

The main result on existence of global action-angle variables is:

Theorem 3.2. *The fibration $\pi: M \rightarrow B$ is topologically trivial if and only if the monodromy and Chern class are trivial. Secondly, the statements (A) and (B) are equivalent.*

(A) *There is a smooth map $(a, \alpha): M \rightarrow \mathbb{R}^n \times (\mathbb{R}/\mathbb{Z})^n$ such that*

$$-\omega = \sum_i d\alpha_i \wedge da_i,$$

- the a_i are constant on the fibers of $\pi: M \rightarrow B$, and
- α is injective on each fiber of $\pi: M \rightarrow B$.

(B) The fibration $\pi: M \rightarrow B$ is topologically trivial and ω is exact.

In the case of mechanical systems, the ω we work with is the canonical area form, so exactness is always guaranteed. It should also be fairly intuitive that the a are our action variables, and the α are our angles. All that is left to is examine the role the Chern class and monodromy play as topological obstructions.

Considering now $F_b := \pi^{-1}(b) \subset M$, we can find a period lattice P_b from the commuting flows of the X_{f_i} . Note that we get these lattices from the invariant tori theorem, which is still applicable. We then take the entire collection of P_b to get another covering P of B . It is well known from covering space theory that we have an action of the fundamental group $\pi(B, b)$ on $P_b \cong H_1(B, \mathbb{Z})$, and should be quick to see that non-trivial monodromy results non-triviality of the bundle structure $M \rightarrow B$.

The Chern class relates to the structure of the canonical area form, and in general the structure of M as a smooth bundle. In particular, the vanishing of the Chern class guarantees the existence of a smooth section $s: B \rightarrow M$. Utilizing the local action-angle coordinates for coordinate charts of M , we see that s corresponds to a map $s|_{B_i}: B_i \rightarrow (\mathbb{R}/\mathbb{Z})^n$, where $\{B_i\}$ is a system of charts for B . All that remains is to check that the $s|_{B_i}$ glue together in a nice way to give a global 2-form, which indeed they do.

We note briefly that, in a paper by Nekhoroshev, it is proved that if B is simply connected and $H^2(B, \mathbb{R}) \cong 0$, then global action-angle coordinates exist. This follows immediately from the theorem in [D1], since vanishing second cohomology of B guarantees that the Chern class is trivial.

We now turn back to our concrete example of the spherical pendulum, which we carried along with us throughout the first section. We have immediately one integral of motion, namely the Hamiltonian

$$H(q^i, p_i, t) = \frac{(p_1)^2}{2} + \frac{(p_2)^2}{2 \sin^2 \phi} + \cos \theta,$$

where $p_1 = \dot{\phi}$ and $p_2 = \dot{\theta} \sin^2(\phi)$.

We have another integral of motion given by the momentum I_z about the z -axis, in the usual xyz -coordinate system of \mathbb{R}^3 . In local coordinates, this is just $I_z = p_2$.

The function $f = (H, I_z)$ is called the momentum map, and is in fact a fibration of T^*S^2 over \mathbb{R}^2 . That T^*S^2 has a trivial Chern class is a straightforward computation given in [D1], so the only possible obstruction to global action-angle coordinates is thus the monodromy. The “quick” argument for this follows from the singular values of E , and the structure of the energy hypersurfaces of $E^{-1}(x)$ for nonsingular $x \in T^2S^2$. This then shows that the fibration $f: f^{-1}(R) \rightarrow R$, where R denotes the set of regular values of f , is non-trivial, so the monodromy cannot be trivial either.

The lengthier computation involves cycles that generate the first homology of each fiber, and showing that monodromy acts on the period lattice of T^*S^2 via the linear transformation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Thus, as above, monodromy is nontrivial and the spherical pendulum does not admit global action-angle coordinates.

4. ALGEBRAIC CONSIDERATIONS

In this final section, we explore some of the more algebraic aspects to classically integrable systems. In particular, we examine more closely the momentum map introduced for the spherical pendulum, the generalizations of this map, and some results along these lines.

Recall the Poisson bracket defined at the beginning of this section, and in particular the fact that it is skew-symmetric and satisfies the Jacobi identity. This gives the space $C^\infty(M)$, all smooth functions on M , the structure of a Lie algebra. Moreover, the map $H \mapsto X_H$ taking a function H to its Hamiltonian vector field gives us a Lie algebra homomorphism from $C^\infty(M)$ to $\Gamma(M)$, the space of vector fields on M with the Lie structure given by the ordinary Lie bracket.

We say that if (M, ω) has a G -action defined on it, then the action is *symplectic* if it preserves the symplectic form, i.e.

$$g^*\omega = \omega, \forall g \in G.$$

In the same way we constructed an action of \mathbb{R}^n on M in Arnold’s invariant tori theorem, we can consider actions of the Lie algebra \mathfrak{g}

on $\mathcal{H}_{\text{loc}}(M)$, the space of *locally Hamiltonian vector fields*. These are precisely the $X \in \Gamma(M)$ such that $i_X\omega$ is closed. Likewise, $\mathcal{H}(M)$ is the space of *Hamiltonian vector fields*, i.e. the $X \in \Gamma(M)$ such that $i_X\omega$ is exact. Note that this extends our consideration of vector fields X_H defined by $i_{X_H}\omega = -dH$. We immediately have the exact sequence

$$\mathcal{H}(M) \longrightarrow \mathcal{H}_{\text{loc}}(M) \longrightarrow H^1(M, \mathbb{R}) \longrightarrow 0.$$

Factoring in the Lie algebra actions of $C^\infty(M)$ and \mathfrak{g} on $\mathcal{H}(M)$ and $\mathcal{H}_{\text{loc}}(M)$ respectively, gives us the diagram

$$\begin{array}{ccccc} C^\infty(W) & \xleftarrow{\tilde{\mu}} & \mathfrak{g} & & \\ \downarrow & & \downarrow & & \\ \mathcal{H}(W) & \longrightarrow & \mathcal{H}_{\text{loc}}(W) & \longrightarrow & H^1(W; \mathbb{R}) \longrightarrow 0 \end{array}$$

The symplectic G -action is called *Hamiltonian* if we can find such a $\tilde{\mu}$ that makes the diagram commute.

Associated to this $\tilde{\mu}$ is the map

$$\mu: M \rightarrow \mathfrak{g}^* = \text{hom}(\mathfrak{g}, \mathbb{R})$$

defined by $x \mapsto (X \mapsto \tilde{\mu}_X(x))$. Note that $\tilde{\mu}_X$ is in $C^\infty(M)$, so the momentum map can be thought of as a smooth function on M . A first, rather straightforward, result is that μ is a Poisson map. This means that, for any two $f, g \in \mathfrak{g}^*$, and any $x \in \mathfrak{g}$,

$$\{f \circ \mu, g \circ \mu\}(x) = \{f, g\}(\mu(x)).$$

Despite the abstract formulations, we can easily connect this map with the momentum map we defined for the spherical pendulum. Recall that the phase space $M = T^*S^2$ of our system had an equivariant n -torus T action, with corresponding Lie algebra \mathfrak{t} . This torus acted on M by the flows of the Hamiltonian vector fields corresponding to our n constants of motion. The momentum map defined at the end of 3.3 is, indeed, a C^∞ function on M , and is invariant under Hamiltonian flow. In particular, it makes the above diagram commute when the torus acts on our flows.

Audin gives a much lengthier, in depth, discussion and treatment of Lie algebra actions on symplectic manifolds in [Au]. It is hoped that our discussion here has whetted the appetite of those interested in exploring this topic further.

5. REFERENCES

- [AM] Arms, J.M., Marsden, J.E., Moncrief, V., *Symmetry and Bifurcations of Momentum Mappings*, 1981
- [Ar] Arnold, V.I., *Mathematical Methods of Classical Mechanics*, 1997
- [Au] Audin, M., *Torus Actions on Symplectic Manifolds*, 2004
- [Ca] Cannas, A., *Lectures on Symplectic Geometry*, 2008
- [Cu] Cushman, R.H., Bates, L.M., *Global Aspects of Classical Integrable Systems*, 1997
- [D1] Duistermaat, J.J., *On Global Action-Angle Coordinates*, 1980
- [D2] Duistermaat, J.J., Heckman, G.J., *On the Variation in the Cohomology of the Symplectic Form of the Reduced Phase Space*, 1982
- [Go] Goldstein, H., *Classical Mechanics*, 1980
- [Gr] Grothendieck, A., *A General Theory of Fibre Spaces With Structure Sheaf*
- [H] Hatcher, A., *Vector Bundles & K-Theory*, 2009
- [M] Michor, P.W., *Momentum Mappings*, 2000
- [MS] Milnor, J.W., Stasheff, J.D., *Characteristic Classes*, 1974
- [S] Spivak, M., *Physics for Mathematicians: Mechanics I*, 2010

6. APPENDIX: SYMPLECTIC MANIFOLDS, GEODESIC FLOW

Symplectic geometry is the study of manifolds equipped with a closed, non-degenerate 2-form. The purpose of this appendix is to give a quick introduction to some topics in symplectic geometry, particularly symplectic forms themselves, symplectomorphisms, Lagrangian submanifolds, and Darboux's theorem, followed by their application to the study of geodesic flow.

6.1. Skew-Symmetric Bilinear Maps. Let V be an m -dimensional real vector space, and $\Omega : V \times V \rightarrow \mathbb{R}$ a bilinear map. Ω is said to be **skew-symmetric** if $\Omega(u, v) = -\Omega(v, u)$ for $u, v \in V$.

Theorem 6.1. *There exists a basis $\mathcal{U} = \{u_1, \dots, u_k, e_1, \dots, e_n, f_1, \dots, f_n\}$ of V such that*

$$\begin{cases} \Omega(u_i, v) = 0, & \text{for each } v \in V, \\ \Omega(e_i, e_j) = \Omega(f_i, f_j) = 0, \\ \Omega(e_i, f_j) = \delta_j^i. \end{cases}$$

With respect to the basis above, we can represent Ω as

$$[\Omega] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \text{Id} \\ 0 & -\text{Id} & 0 \end{bmatrix}.$$

Proof. The proof is just a Gram-Schmidt sort of process, which we give a sketch of:

- Define $U = \{u \in V \mid \Omega(u, v) = 0 \text{ for all } v \in V\}$. Choose a basis $\{u_1, \dots, u_k\}$ of U , and a complementary space $W \subset V$ such that $V = U \oplus W$.
- Take any nonzero $e_1 \in W$. Since $e_1 \notin U$, there is some $f_1 \in W$ such that $\Omega(e_1, f_1) \neq 0$, and we can rescale f_1 so that $\Omega(e_1, f_1) = 1$. Define $W_1 = \langle e_1, f_1 \rangle$, i.e. the span of e_1 and f_1 , and let $W_1^\Omega = \{w \in W \mid \Omega(w, v) = 0 \text{ for all } v \in W_1\}$. W_1^Ω is called the **symplectic orthogonal** to W_1 .
- We have two facts about W_1 and W_1^Ω : $W_1 \cap W_1^\Omega = \{0\}$ and $W = W_1 \oplus W_1^\Omega$. These are quick calculations and left to the reader (they are also carried out in [Ca]).

- We now pick a nonzero $e_2 \in W_1^\Omega$ and a $f_2 \in W_1^\Omega$ such that $\Omega(e_2, f_2) = 1$. Define $W_2 = \langle e_2, f_2 \rangle$, W_2^Ω the orthogonal complement, etc.
- Since V is finite dimensional, this process eventually stops, and we have

$$V = U \oplus W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

By construction, and a quick mental check, Ω has the desired properties with respect to this basis.

□

U doesn't depend on the choice of basis, so k is an invariant of the pair (V, Ω) . Similarly, since $k + 2n = \dim V$, we see that n is an invariant of (V, Ω) as well. We call n the **rank** of Ω .

A **symplectic map** is a skew-symmetric, bilinear, nondegenerate map. From the construction of \mathcal{U} , this means that $k = 0$. Another equivalent definition is to say that the map $\bar{\Omega} : V \rightarrow V^*$ defined by $\bar{\Omega}(v) = \Omega(v, \cdot)$ is bijective. When this happens, we call (V, Ω) a **symplectic vector space**, and \mathcal{U} is the corresponding **symplectic basis**. Note that a symplectic vector space must have even dimension.

The prototypical example of a symplectic vector space is \mathbb{R}^{2n} , with $e_i = (0, \dots, 1, \dots, 0, \dots, 0)$ having a 1 in the i th slot and zeros elsewhere, and $f_i = (0, \dots, 0, 0, \dots, 1, \dots, 0)$ having a 1 in the $(n+i)$ th slot and zeros elsewhere. Ω is then defined in a natural way on these basis vectors.

6.2. Symplectic Manifolds. Now let M be a smooth manifold and ω a 2-form on M .

Definition 6.2. The form ω is **symplectic** if it is closed and $\omega_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is symplectic for each $p \in M$.

Definition 6.3. A **symplectic manifold** is a pair (M, ω) , where M is a manifold and ω is a symplectic form on M .

Let $M = \mathbb{R}^{2n}$ with coordinates $x_1, \dots, x_n, y_1, \dots, y_n$. The prototypical example of a symplectic manifold is (M, ω) , where

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i$$

and on each $T_p M$ we have the symplectic basis

$$\mathcal{U} = \{\partial_{x^1}, \dots, \partial_{x^n}, \partial_{y^1}, \dots, \partial_{y^n}\}.$$

Just as maps between Riemannian manifolds that preserve the metric structure are called isometries, maps between symplectic manifolds that preserve the symplectic structure are called symplectomorphisms. Formally,

Definition 6.4. Let (M_1, ω_1) and (M_2, ω_2) be two symplectic manifolds. Then a diffeomorphism $\phi : M_1 \rightarrow M_2$ is called a **symplectomorphism** if $\phi^* \omega_2 = \omega_1$.

We can now state Darboux's theorem, which gives us the local structure of any symplectic manifold:

Theorem 6.5 (Darboux's Theorem). *Let (M, ω) be a $2n$ -dimensional symplectic manifold, and let $p \in M$. Then there is a local coordinate chart $\mathcal{U} = \{x^1, \dots, x^n, y^1, \dots, y^n\}$ of p such that, in \mathcal{U} ,*

$$\omega = \sum_{i=1}^n dx^i \wedge dy^i.$$

In other words, any symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n}, \omega)$.

The chart \mathcal{U} above is called a **Darboux chart**.

One proof of Darboux's theorem requires a result due to Jurgen Moser:

Theorem 6.6 (The Moser Relative Theorem). *Let M be a manifold, X a compact submanifold of M , and $i : X \hookrightarrow M$ the inclusion map. If ω_0 and ω_1 are two symplectic forms on M , and $\omega_0|_p = \omega_1|_p$ for all $p \in X$, then there exist neighborhoods \mathcal{U}_0 and \mathcal{U}_1 of X , and a diffeomorphism $\phi : \mathcal{U}_0 \rightarrow \mathcal{U}_1$ such that $\phi^* \omega_1 = \omega_0$ and $\phi \circ i_0 = i_1$, where $i_j : X \hookrightarrow \mathcal{U}_j$.*

Proof of Darboux's Theorem. We use Moser's relative theorem, with $X = \{p\}$. The idea is that we have two symplectic forms at p , the original symplectic form ω , and another form $\omega_1 = \sum_{i=1}^n dx'_i \wedge dy'_i$, which can always be constructed using the symplectic basis of $T_p M$ (we are just representing ω in a new basis). Now, by Moser's theorem, we can find neighborhoods \mathcal{U}_0 and \mathcal{U}_1 of p , and a diffeomorphism $\phi : \mathcal{U}_0 \rightarrow \mathcal{U}_1$

such that $\phi(p) = p$ and $\phi^*(\omega_1) = \omega_0$. In other words,

$$\begin{aligned}\omega = \omega_0 &= \phi^*\left(\sum_{i=1}^n dx'_i \wedge dy'_i\right) \\ &= \sum_{i=1}^n d(x'_i \circ \phi) \wedge d(y'_i \circ \phi).\end{aligned}$$

Since ϕ is a diffeomorphism, defining $x_i = x'_i \circ \phi$ gives us the Darboux chart around p . \square

Two quick remarks regarding Darboux's theorem:

- Darboux's theorem tells us that symplectic manifolds are flat, i.e. have no curvature, since they all look like \mathbb{R}^{2n} locally.
- Symplectic manifolds must be even dimensional. S^1 , a manifold, does not admit a symplectic structure. TS^1 , the tangent bundle, does admit a symplectic structure though. In general, we can expect TM and T^*M to admit symplectic structures even if M doesn't.

6.3. Lagrangian Submanifolds. “Zero spaces” are important objects of study in many areas of math: ideals in ring theory, kernels of linear maps, etc. Lagrangian submanifolds are, in a sense, the zero spaces of symplectic geometry.

Definition 6.7. A **lagrangian subspace** (Y, Ω) of a symplectic vector space (V, Ω) is a subset $Y \subset V$ such that $\dim Y = \frac{1}{2} \dim V$ and $Y = Y^\Omega$, i.e. $\Omega|_{Y \times Y} \equiv 0$. A **lagrangian submanifold** (Y, ω) of a symplectic manifold (M, ω) is a submanifold $Y \subset M$ such that $T_p Y$ is a lagrangian subspace of $T_p M$ for every $p \in Y$. Equivalently, if $i : Y \hookrightarrow M$ is the inclusion map, Y is called **lagrangian** if $i^* \omega = 0$ and $\dim Y = \frac{1}{2} \dim M$.

One immediate use of lagrangian subspaces is to be able to tell whenever a diffeomorphism of smooth manifolds preserves the symplectic structures, i.e. when is a diffeomorphism a symplectomorphism? To answer this, we need the appropriate framework.

Let (M_1, ω_1) and (M_2, ω_2) be two $2n$ -dimensional symplectic manifolds, and let $\phi : M_1 \rightarrow M_2$ be a diffeomorphism. Given the product space $M_1 \times M_2$, we have two projection maps $\text{pr}_1 : M_1 \times M_2 \rightarrow M_1$ and $\text{pr}_2 : M_1 \times M_2 \rightarrow M_2$ defined in the obvious ways. We note that

$$\omega = \lambda_1(\text{pr}_1)^* \omega_1 + \lambda_2(\text{pr}_2)^* \omega_2, \quad \lambda_i \in \mathbb{R},$$

is a symplectic (closed and nondegenerate) form on $M_1 \times M_2$. Specializing to $\lambda_1 = 1, \lambda_2 = -1$ gives us the **twisted product form** on $M_1 \times M_2$:

$$\tilde{\omega} = (\text{pr}_1)^*\omega_1 - (\text{pr}_2)^*\omega_2.$$

We also need the graph of the diffeomorphism ϕ , which is defined to be

$$\Gamma_\phi = \{(p, \phi(p)) | p \in M_1\} \subset M_1 \times M_2.$$

We have the following

Proposition 6.8. *A diffeomorphism ϕ is a symplectomorphism if and only if Γ_ϕ is a lagrangian submanifold of $(M_1 \times M_2, \cdot)$.*

Proof. Let $\gamma : M_1 \rightarrow M_1 \times M_2$ be defined by $p \mapsto (p, \phi(p))$. Then the graph Γ_ϕ is lagrangian if and only if $\gamma^*\tilde{\omega} = 0$. We see, though, that

$$\begin{aligned} \gamma^*\tilde{\omega} &= \gamma^*\text{pr}_1^*\omega_1 - \gamma^*\text{pr}_2^*\omega_2 \\ &= (\text{pr}_1 \circ \gamma)^*\omega_1 - (\text{pr}_2 \circ \gamma)^*\omega_2. \end{aligned}$$

Since $\text{pr}_1 \circ \gamma = 1_{M_1}$ and $\text{pr}_2 \circ \gamma = \phi$, we conclude that $\gamma^*\tilde{\omega} = 0$ if and only if $\phi^*\omega_2 = \omega_1$, i.e. ϕ is a symplectomorphism. \square

The remainder of this appendix is devoted to proving two results in symplectic geometry related to Riemannian geometry and Hamiltonian mechanics. The first relates to geodesic flow, and how the symplectic structure of the cotangent bundle gives us our familiar geodesic equation. The second result is Liouville's theorem, which states that the volume form on symplectic manifolds is preserved under phase (vector) flow. In fact, we prove a stronger result due to V.I. Arnold from the 1960s, which states that the symplectic form is preserved under vector flow.

6.4. The Cotangent Bundle. Given a smooth manifold M , the cotangent bundle T^*M is defined to be the set of all linear functionals on the tangent bundle TM . Fix a point $x \in M$, and consider a coordinate chart $\{U, \phi, \{q^i\}\}$, where U is a neighborhood of x , ϕ is a diffeomorphism $\phi: U \rightarrow \mathbb{R}^n$, and $q^i, i = 1, \dots, n$ are the local coordinates. Then we have a basis for the tangent space $T_x M$ given by $\{\partial_{q^i}\}$. From this we can construct a basis for the dual space $T_x^* M$, arguably the simplest of which is $\{dq^i\}$, where $dq^i(\partial_{q^j}) = \delta_{ij}$. Since a form $\xi \in T_x^* M$ can be written as $\xi = p_i dq^i$, we have a natural coordinate system for T^*U :

$$(x, \xi) \mapsto (q^1, \dots, q^n, p_1, \dots, p_n).$$

In this way, we see that T^*M is a $2n$ -manifold, with local charts given by the above map. It is a quick calculation that on intersections of charts the transition functions are smooth.

There are two forms on T^*M we will be especially interested in: the Liouville 1-form and the canonical symplectic form. As above, we fix a point $x \in M$, a local chart $\{U, \phi, \{q^i\}\}$ of x , and local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$ of T^*U . The form $\lambda = p_i dq^i$ is called the **Liouville 1-form** or **tautological 1-form**. Note that this form is defined on T^*U , not just T_x^*M . The form $\omega = -d\lambda = dq^i \wedge dp_i$ is called the **canonical symplectic form** on T^*U . One can check explicitly that this form is invariant under change of coordinates, though we can also give a coordinate free definition (which may be harder to visualize).

Let $\pi: T^*M \rightarrow M$ be the natural projection, let $\xi \in T_x^*M, x \in M$, and set $y = (x, \xi)$. The **coordinate free tautological 1-form** λ' is defined pointwise as

$$\lambda'_y = (d\pi_y)^* \xi \in T_y^*(T^*M).$$

To make this a little clearer, set $M' = T^*M$. Then since $\pi: M' \rightarrow M$, we have $d\pi_y: T_y M' \rightarrow T_x M$, and thus $(d\pi_y)^*: T_x^*M \rightarrow T_y^* M'$. For $v \in T_y M' = T_y(T_x^*M)$, we have

$$\lambda'_y(v) = \xi((d\pi_y)v).$$

In this way we can define λ' on all of T^*M , and a nice exercise is to verify that, in local coordinates, $\lambda' = \lambda$.

6.5. Generating Functions. Let (N, ω) be a symplectic manifold, and $H: N \rightarrow \mathbb{R}$ be some smooth function. In the case that $N = T^*M$ for some smooth n -manifold M , we think of H as a function of $2n$ -variables, $H = H(q^1, \dots, q^n, p_1, \dots, p_n)$. We then have

$$dH = \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i.$$

Likewise, the form ω defines a map $\tilde{\omega}: TM \rightarrow T^*M$, defined by $\tilde{\omega}(v) = \omega(v, \cdot)$. Since ω is symplectic, $\tilde{\omega}$ is an isomorphism. Thus, we can find a vector field X_H on T^*M such that $\tilde{\omega}(X_H) = dH$, which we may also write as $i_{X_H} \omega$. This vector field X_H is called the **Hamiltonian vector field** with corresponding **Hamiltonian** H .

Now fix M a smooth manifold, T^*M the cotangent bundle, and consider a local chart with coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$. Using this chart, we construct the canonical symplectic form $\omega = dq^i \wedge dp_i$. Now

any vector field X on T^*M (so $X : T^*M \rightarrow T(T^*M)$ given by $(x, \xi) \mapsto X_{(x, \xi)}$) can be written as $X = a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}$, where $a^i, b_i : T^*M \rightarrow \mathbb{R}$. Recalling that $i_X(dq^i \wedge dp_i) = (i_X dq^i) \wedge dp_i - dq^i \wedge (i_X dp_i)$ (where $i_X dq^i = dq^i(X)$, for example), we compute

$$\begin{aligned} i_X \omega &= (dq^i)(a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}) \wedge dp_i - dq^i \wedge (dp_i)(a^i \frac{\partial}{\partial q^i} + b_i \frac{\partial}{\partial p_i}) \\ &= a^i dp_i - b_i dq^i. \end{aligned}$$

When is X a Hamiltonian vector field? Precisely when $dH = i_X \omega$ for some $H : T^*M \rightarrow \mathbb{R}$, i.e.

$$\begin{aligned} dH &= i_X \omega, \\ \frac{\partial H}{\partial q^i} dq^i + \frac{\partial H}{\partial p_i} dp_i &= a^i dp_i - b_i dq^i, \\ 0 &= \left(\frac{\partial H}{\partial q^i} + b_i \right) dq^i + \left(\frac{\partial H}{\partial p_i} - a^i \right) dp_i. \end{aligned}$$

Since this must hold true on all of T^*M , we conclude that X is the Hamiltonian flow if and only if the **Hamiltonian equations** are satisfied:

$$\begin{cases} a^i &= \frac{\partial H}{\partial p_i}, \\ b_i &= -\frac{\partial H}{\partial q^i}. \end{cases}$$

We now specialize to the case when X is a tangent vector field along a path $c : I \rightarrow T^*M$, so that $c(t) = (q^1(t), \dots, q^n(t), p_1(t), \dots, p_n(t))$ and

$$\begin{aligned} X &= \frac{dq^i(t)}{dt} \frac{\partial}{\partial q^i} + \frac{dp_i(t)}{dt} \frac{\partial}{\partial p_i} \\ &= \dot{q}^i(t) \frac{\partial}{\partial q^i} + \dot{p}_i \frac{\partial}{\partial p_i}. \end{aligned}$$

Hamilton's equations then tell us that

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i}, \\ \dot{p}_i &= -\frac{\partial H}{\partial q^i}, \end{aligned}$$

which should be familiar from any intermediate mechanics class.

6.6. Geodesic Flow. Let γ be a geodesic on M with initial conditions $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. The **geodesic flow** Φ_t of γ is defined as $\Phi_t(x, v) = (\exp(tv), \frac{d\exp(tv)}{dt})$. In fact, we will see that we can arrive at the geodesic flow by an appropriate choice of Hamiltonian H .

Let (M, g) be a Riemannian manifold with $x \in M$. In local coordinates we can write g_{ij} , which in turn gives the inverse metric g^{ij} that acts on 1-forms. How so? Since $g_x: T_x M \times T_x M \rightarrow \mathbb{R}$, we can turn this into a map $\tilde{g}_x: T_x M \rightarrow T_x^* M$. By nondegeneracy \tilde{g}_x has an inverse $\tilde{g}_x^{-1}: T_x^* M \rightarrow T_x M$, which we then convert to a function $g_x^{-1}: T_x^* M \times T_x^* M \rightarrow \mathbb{R}$.

Let $(x, \xi) \in T^* M$ so that $x = (q^1, \dots, q^n)$ and $\xi = p_i dq^i$. and let $H: T^* M \rightarrow \mathbb{R}$ be defined by

$$\begin{aligned} H(x, \xi) &= \frac{1}{2} g_x^{-1}(\xi, \xi) \\ &= \frac{1}{2} g^{ij}(x) p_i p_j. \end{aligned}$$

To determine the Hamiltonian flow of H , we utilize Hamilton's equations:

$$\begin{aligned} \dot{q}^i &= \frac{\partial H}{\partial p_i} = g^{ij}(x) p_j, \\ \dot{p}_k &= -\frac{\partial H}{\partial q^k} = -\frac{1}{2} g^{ij}_{,k}(x) p_i p_j. \end{aligned}$$

Since $g^{ij} g_{jl} = \delta_l^i$, and differentiating with respect to k , shows that

$$\begin{aligned} g^{ij}_{,k} g_{jl} + g^{ij} g_{jl,k} &= 0 \\ g^{im}_{,k} &= -g^{ij} g^{lm} g_{jl,k}. \end{aligned}$$

Differentiating the first of Hamilton's equations with respect to time gives

$$\ddot{q}^i = g^{ij}_{,k}(x) \dot{q}^k p_j + g^{ij}(x) \dot{p}_j.$$

Substituting in for \dot{q}^k, p_j , and \dot{p}_j then results in the familiar geodesic equation

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = 0.$$

In this way, finding the vector flow corresponding to the energy function on $T^* M$ results in geodesics. Note that these calculations were all local, i.e. in an infinitesimal neighborhood of x . There are other formulations that give a more global construction, utilizing the metric d on (M, g) induced by geodesics.

In [Ca], a more general method of constructing these kinds of symplectomorphisms is shown. Given cotangent bundles $M_1 = T^*X_1, M_2 = T^*X_2$, we want to construct a symplectomorphism $\phi: M_1 \rightarrow M_2$. By the last theorem of the last lecture, a diffeomorphism is a symplectomorphism if and only if its graph is a Lagrangian submanifold of the product space with the twisted form. Thus, we start with any Lagrangian submanifold of $M_1 \times M_2$, “twist” it to obtain another Lagrangian submanifold, and then check whether the twisted space is the graph of a symplectomorphism.

The idea is that given two spaces X_1, X_2 , and a smooth function $f: X_1 \times X_2 \rightarrow \mathbb{R}$, we can look at the Lagrangian submanifold $Y = \{(x, y), df_{(x,y)} | (x, y) \in X_1 \times X_2\}$ (check that this is indeed a Lagrangian submanifold). Equivalently, this can be thought of as an equivalent product space $Y = \{(x, y, df_x, df_y) | x \in X_1, y \in X_2\}$ where $df_x \in T_x^*X_1$ and $df_y \in T_y^*X_2$. “Twist” this space to get $Y^\sigma = \{(x, y, df_x, -df_y) | x \in X_1, y \in X_2\}$. If this space is the graph of some diffeomorphism $\phi: T^*X_1 \rightarrow T^*X_2$, then ϕ is also a symplectomorphism, and is said to be generated by f . If such a ϕ exists, we must have $\phi(x, \xi) = (y, \eta)$, such that $\xi = df_x \in T_x^*X_1$ and $\eta = -df_y \in T_y^*X_2$, Hamilton’s equations.

If M is an n -manifold, we can find the geodesic flow between two points by letting $f: M \times M \rightarrow \mathbb{R}$ be defined by $f(x, y) = \frac{1}{2}d(x, y)^2$, where d is the metric induced by geodesics (the length of the minimizing geodesic between x and y).

6.7. Liouville’s Theorem. Liouville’s theorem is a classical theorem of Hamiltonian mechanics, and concerns the behaviour of mechanical systems under phase flow. Let (M, ω) be a symplectic manifold (so that $\dim M = 2n$). If we have local coordinates $(q^1, \dots, q^n, p_1, \dots, p_n)$, then locally we can write $\omega = dq^i \wedge dp_i$. Moreover,

$$\omega^n = (\pm 1) dq^1 \wedge \dots \wedge dq^n \wedge dp_1 \wedge \dots \wedge dp_n$$

is a volume form on M . Liouville’s theorem states that

Theorem 6.9. *Phase flow (mechanical motions) in M preserves ω^n .*

The usual setting for Liouville’s theorem is, as mentioned above, Hamiltonian mechanics. The manifold M is actually the cotangent space to some n -manifold X , where X is the configuration space of a mechanical system and $M = T^*M$ is the set of all possible configurations of the system, with accompanying momenta.

In fact, a much stronger statement holds than Liouville's theorem. In the 1960s, V.I. Arnold showed that Hamiltonian phase flow preserves the symplectic form. It should be clear that Liouville's theorem is a corollary of this, since if some vector flow preserves ω , then it preserves ω^n as well by the properties of pullbacks.

Let (M, ω) be a symplectic manifold, and $H: M \rightarrow \mathbb{R}$ some smooth function. As above, we define X_H as the Hamiltonian vector field corresponding to H , i.e. the vector field such that $i_{X_H}\omega = dH$ holds. Let $\rho^t: M \rightarrow M$ be a one parameter group of diffeomorphisms generated by X_H . Arnold's theorem states that

Theorem 6.10.

$$(\rho^t)^*\omega = \omega, \quad \forall t.$$

The proof, which can be found in full detail in [Ar] section 38, is a simple calculation using Stokes' theorem and certain chains in M : let c be a 1-chain in M , and let Jc be the “track” of the chain c under the homotopy ρ^t for $0 \leq t \leq \tau$; essentially Jc is the chain swept out by c through ρ^t . One can easily verify that $\partial(Jc) = \rho^\tau c - c - J(\partial c)$.

A short lemma that is needed, which I will not prove but can be found in [Ar], is that $\int_{J(\partial c)} \omega = 0$. With all these considerations in hand, we can proceed with the proof.

Proof.

$$0 = \int_{Jc} d\omega = \int_{\partial Jc} \omega = \left(\int_{\rho^\tau c} - \int_c - \int_{J\partial c} \right) \omega = \int_{\rho^\tau c} \omega - \int_c \omega,$$

whereby we used the closedness of ω in the first equality and Stokes' theorem in the second equality. \square

Another proof is given in [Ca]. While much shorter, it doesn't give as much geometric insight into the behaviour of the phase flow. For the proof, we recall two facts:

- Cartan's formula tells us how to evaluate the Lie derivative on forms, i.e. if X is a vector field and ω is some form, then

$$\mathcal{L}_X \omega = i_X d\omega + d(i_X \omega).$$

- Since ρ^t is the group of diffeomorphisms generated by X_H , a Hamiltonian vector field, we have that $\rho^0 = Id_M$ and $\frac{d\rho^t}{dt} = X_H|_{\rho^t}$.

Second proof.

$$\begin{aligned}
 \frac{d}{dt}(\rho^t)^*\omega &= \mathcal{L}_{X_H}\omega \\
 &= d(i_{X_H}\omega) + i_{X_H}d\omega \\
 &= (d(dH) + i_{X_H} \circ 0) = 0,
 \end{aligned}$$

so $(\rho^t)^*\omega$ is constant for every t .

□

7. APPENDIX: CHARACTERISTIC CLASSES

Characteristic classes are an important area of study, both in pure and applied contexts. In this appendix, we develop the necessary foundation to define the Pontrjagin and Chern classes of a fibre bundle E over a smooth manifold M , and give an application of Chern classes to physics.

7.1. Fibre Bundles. We start with a definition, followed by a few examples.

Definition 7.1. Let E, B be smooth manifolds with a map $\pi : E \rightarrow B$, and let F be another smooth manifold. Then $\zeta = (E, \pi, B, F)$ is called a *fibre bundle* if it satisfies the *local triviality condition*: given a point $b \in B$ and a neighborhood U of b , we have a homeomorphism $\phi : \pi^{-1}(U) \rightarrow U \times F$ such that $\pi = \pi_1 \circ \phi : \pi^{-1}(U) \rightarrow U \times F \rightarrow U$. E is called the *total space*, B is called the *base space*, F is called the *fibre*, and π is called the *projection*. In particular, for each b we have $E_b = \pi^{-1}(b) \cong F$. Relaxing the restriction that all fibres be homeomorphic to F to homotopically equivalent to F , we arrive at the notion of a fibration.

We will often denote a fibre bundle by writing $\pi : E \rightarrow B$. The idea of a fibre bundle is to parameterize a collection of manifolds by another manifold. The examples below should help concretize this idea:

Example 7.2 (Example: Trivial Bundle). The space $E = B \times F$ is clearly a fibre bundle, with $\pi(b, f) = b$ for $b \in B, f \in F$. An easy example to visualize is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$, with the base space the horizontal axis, and the fibres each constant line. The projection is given by $\pi(x, y) = x \sim (x, 0)$. Similarly, the cylinder $S^1 \times \mathbb{R}$ is a trivial bundle with $\pi(\theta, x) = \theta$ for $\theta \in S^1, x \in \mathbb{R}$.

Example 7.3 (Example: Tangent Bundle). Let M be a smooth n -dimensional manifold, and let $p \in M$. Then we can construct the tangent space $T_p M$ as the vector space of all tangent vectors to M at p . We then define the tangent bundle as $TM := \cup_{p \in M} T_p M$. This is a fibre bundle with total space TM , base space M , and fibres $T_p M \cong \mathbb{R}^n$. For $(p, u) \in T_p M$, the projection is simply $\pi(p, u) = p$. It is a standard result that this space satisfies the local triviality condition.

Example 7.4 (Example: Möbius Band). Consider now the Möbius band $= M$, the strip $[0, 1] \times (-1, 1)$ where we associate $(0, y) \sim (1, -y)$. Notice that the set of points $(x, 0), x \in I$ form a circle S^1 that wraps

around the center of M . If we take any subset $U \subsetneq S^1 \hookrightarrow M$, then $U \times (-1, 1)$ is a local trivialization of M , so indeed M can be thought of as a fibre bundle. In this case, M is the total space, S^1 is the base space, the fibres are $F = (-1, 1)$, and the projection is simply $\pi(x, y) = x$. Notice that this space is not homeomorphic to $S^1 \times (-1, 1)$, since M is, in a sense, twisted while $S^1 \times (-1, 1)$ is not. In fact, characteristic classes give one way of measuring the extent to which a given fibre bundle is not trivial.

The first two examples were both vector bundles, since the fibres each had a natural vector space structure. The Möbius band, however, is not a vector bundle, since $(-1, 1)$ does not have a natural vector space structure. We can give it such a structure though by either specifying such a structure on $(-1, 1)$, or taking some homeomorphism $(-1, 1) \rightarrow \mathbb{R}$ and then working with the natural vector structure of \mathbb{R} .

In what follows, we will assume all fibres are vector spaces. Two definitions of characteristic classes will be given: the first through the curvature form associated to a connection on a manifold, and the second in purely cohomological terms.

7.2. Sections. Since characteristic classes arise as certain invariants related to curvature forms, which in turn rely on the notion of a connection, we will develop the necessary terminology in this section. We start by defining sections:

Definition 7.5. A *section* of a vector bundle $\pi : E \rightarrow M$ is a smooth function $s : M \rightarrow E$ such that $\pi \circ s = \text{Id}_M$.

Informally, at each point $p \in M$, a section gives us a choice of vector $s(p) \in \pi^{-1}(p)$ that varies smoothly with respect to p . As a (trivial) example, consider the zero section defined by $s(p) = 0 \in \pi^{-1}(p)$ for each $p \in M$. We can also rephrase the local trivialization condition that vector bundles satisfy in terms of sections.

Definition 7.6. Let $U \subset M$ be an open set, where M is n -dimensional. A *frame field* over U is a collection of sections $s_i : U \rightarrow E, i = 1, \dots, n$, such that $s_1(p), \dots, s_n(p)$ span $\pi^{-1}(p)$ for every $p \in U$.

A local trivialization of a subset $\phi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ is equivalent to finding a frame field for that set, since any point in $U \times \mathbb{R}^n$ can be expressed as $(p, a^1 s_1(p), \dots, a^n s_n(p))$. We denote the collection of all sections of a vector bundle by $\Gamma(E)$, and briefly note that $\Gamma(E)$ can be

made into a vector space by defining addition and scalar multiplication pointwise: for $s, s' \in \Gamma(E)$, $a \in \mathbb{R}$, we define

$$(s + s')(p) = s(p) + s'(p), \quad (as)(p) = as(p).$$

Example 7.7 (Example: Sections of a tangent bundle). Consider a manifold M , and its tangent bundle TM . Then $\Gamma(TM)$ is the collection of all possible vector fields on M , since a section of a tangent bundle is just a vector field on the base manifold. We use the standard notation $\Gamma(TM) = \mathfrak{X}(M)$.

As a quick aside, we need to be able to examine how local trivializations near each other are related. To be more precise, let $\pi : E \rightarrow M$ be a vector bundle, and let U_a, U_b be two subsets of M with local trivializations $\phi_a : \pi^{-1}(U_a) \rightarrow U_a \times \mathbb{R}^n$ and $\phi_b : \pi^{-1}(U_b) \rightarrow U_b \times \mathbb{R}^n$. Now fix some $p \in U_a \cap U_b$, and consider the composite map $\phi_a \circ \phi_b^{-1} : (U_a \cap U_b) \times \mathbb{R}^n \rightarrow (U_a \cap U_b) \times \mathbb{R}^n$. We can write this composition as $\phi_a \circ \phi_b^{-1}(p, v) = (p, g_{ab}(p)v)$, where $g_{ab} : U_a \cap U_b \rightarrow GL(n, \mathbb{R})$. This is because ϕ_a, ϕ_b are homeomorphisms, so are invertible, from which we conclude that $\phi_a \circ \phi_b^{-1}(p, \cdot)$ is an invertible linear operator on an n -dimensional real vector space, i.e. $g_{ab}(p) \in GL(n, \mathbb{R})$. These g_{ab} are the *transition functions* of the trivializations ϕ_a, ϕ_b . If U_c is another subset of M , a quick calculation shows that $g_{ab}g_{bc} = g_{ac}$ on $U_a \cap U_b \cap U_c$, which we call the cocycle condition.

7.3. Connections. Connections are the natural generalization of covariant derivatives from analysis in \mathbb{R}^n :

Definition 7.8. A *connection* ∇ on a vector bundle $\pi : E \rightarrow M$ is a bilinear map

$$\nabla : \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)$$

that satisfies the following two properties:

- (i) $\nabla_{fX}s = f\nabla_Xs$,
- (ii) $\nabla_X(fs) = f\nabla_Xs + (Xf)s$,

where $f \in C^\infty(M)$, $X \in \mathfrak{X}(M)$, and $s \in \Gamma(E)$. The second property above can be thought of as a kind of Liebnitz rule for connections.

All vector bundles admit connections, and in fact can admit a great number of them. For simplicity, we will work just with the product bundle $M \times \mathbb{R}^n$, but it should be clear how this case is used to treat the general situation. In particular, we take an open cover of M , perform

construct local (trivial) connections on each local trivialization, and glue these connections together by a partition of unity on M .

We now return to $M \times \mathbb{R}^n$. Let $U \subset M$ have local coordinates (x^1, \dots, x^n) , so that the fibre \mathbb{R}^n is generated by the vectors $\partial_1, \dots, \partial_n$, where $\partial_i := \frac{\partial}{\partial x^i}$. Define a connection ∇_{∂_i} by $\nabla_{\partial_i} \partial_j = 0$ for all i, j . The Liebnitz rule above shows us that for an arbitrary vector $X = \sum_i x^i \partial_i$, we get

$$\nabla_{\partial_i} X = \sum_j (\partial_i x^j) \partial_j,$$

which is just the covariant derivative of the vector (x^1, \dots, x^n) in the i -th direction. Derivations are thus a natural generalization of the familiar partial derivatives in \mathbb{R}^n . This connection is called the trivial connection, and it should be easy to see how to define a connection on the whole manifold by patching together trivial connections in each coordinate patch using a partition of unity.

7.4. Curvature. Curvature is a measure of how curved a surface is: for curves it is just the magnitude of the acceleration vector, for surfaces it is the product of the two principal sectional curvatures. It isn't so clear, however, how to generalize this notion to arbitrary smooth manifolds. The following definition will give a generalization, which can be seen to coincide with the standard definitions of curvature for curves and surfaces:

Definition 7.9. Let $\pi : E \rightarrow M$ be a vector bundle, and ∇ a connection on M . We define the *curvature* associated to this connection by

$$R(X, Y) = \frac{1}{2}(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X, Y]})$$

for vector fields $X, Y \in \mathfrak{X}(M)$.

Indeed, curvature as it is defined here isn't a scalar, but rather an operator on $\Gamma(E)$. A quick check shows that $R(fX, gY)(hs) = fghR(X, Y)(s)$ for functions f, g, h and section s . We also have $R(X, Y) = -R(Y, X)$. Now, fix an open set $U \subset M$ and fix a frame field s_1, \dots, s_n on U . We can then write

$$R(X, Y)(s_j) = \sum_i \Omega_j^i(X, Y) s_i$$

since the s_i span the fibre at each point in U . Since R is bilinear and alternating in its arguments, we conclude that the Ω_j^i are also alternating and bilinear, and so in fact are 2-tensors. Writing $\Omega = (\Omega_j^i)$ for the $n \times n$ matrix of Ω_j^i s, we get a matrix valued 2-form Ω , called the curvature form.

Finally, let U_a, U_b be two subsets with local trivializations, and let Ω_a, Ω_b be the curvature forms associated to U_a, U_b respectively. Then a quick computation will show that $\Omega_a = g_{ab}^{-1} \Omega_b g_{ab}$, where the g_{ab} were transition functions detailed above.

7.5. Pontrjagin and Chern Classes. Pontrjagin classes are invariant polynomials associated to curvature forms. We call a real valued polynomial function f , with arguments real $n \times n$ matrices, *invariant* if $f(X) = f(A^{-1}XA)$ whenever $A \in GL(n, (R))$. An elementary result in algebra states that the ring of invariant polynomials is isomorphic to the ring of symmetric polynomials, where the symmetric polynomials are those polynomials invariant under permutation of indices. If we let $N = \{1, \dots, n\}$, $J = \{j_1, \dots, j_i\} \subset N$, then the ring of symmetric polynomials is generated by $\sigma_i = \sum_{J \subset I} x_{j_1} \cdots x_{j_i}$.

Since curvature forms transform in a nice way, i.e. by $GL(n, \mathbb{R})$, we see that $f(\Omega_a) = f(\Omega_b)$ whenever $U_a \cap U_b \neq \emptyset$ and f is an invariant polynomial. Notice that if f has degree k , then $f(\Omega)$ is a $2k$ -form. It is a nontrivial assertion that $f(\Omega)$ is a closed form, and the proof comes down to a direct computation and utilization of Bianchi's identity ($d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$). This allows us to consider the class $[f(\Omega)] \in H_{DR}^{2k}(M)$, the $2k$ th De Rham cohomology group. What one finds, though, is that first $[f(\Omega)]$ does not depend on the choice of ∇ . The idea for the proof of this is, given two connections ∇_0, ∇_1 on $\pi : E \rightarrow M$, we define a new connection $\bar{\nabla}$ on the new vector bundle $\pi \times Id : E \times \mathbb{R} \rightarrow M \times \mathbb{R}$ in such a way that $\bar{\nabla}$ is a homotopy from ∇_0 to ∇_1 . Having done this, we immediately get that $[f(\Omega_0)] = [f(\Omega_1)]$, where Ω_i is the curvature form of ∇_i . Thus, the cohomology classes of curvature forms do not depend on the choice of connection, but rather the total space E . Second, $[f(E)] = 0$ whenever f has odd degree, which follows by defining a Riemannian metric in E , and a connection on E that satisfies

$$X\langle s, s' \rangle = \langle \nabla_X s, s' \rangle + \langle s, \nabla_X s' \rangle,$$

where $X \in \mathfrak{X}(M)$ and $s, s' \in \Gamma(E)$. Having done this, we find that the curvature form Ω is a skew-symmetric matrix. For k odd, Ω^k is also a skew symmetric matrix, so $\text{Tr}(\Omega^k) = 0$. The rest follows from

some more elementary results on symmetric polynomials, in particular Newton's formula. As a result, we denote these classes by $[f(E)] \in H_{DR}^{2k}(M)$ and call them the *characteristic class* of E corresponding to f .

We can now give the following

Definition 7.10. *Pontrjagin classes* are the characteristic classes associated to the invariant polynomials $\frac{1}{(2\pi)^{2k}}\sigma_{2k}$. We write these as

$$p_k(E) = \left[\frac{1}{(2\pi)^{2k}} \sigma_{2k}(E) \right] \in H_{DR}^{4k}(M).$$

We also have the *total Pontrjagin class*

$$p(E) = 1 + p_1(E) + \cdots + p_{\lfloor n/2 \rfloor}(E) \in H_{DR}^*(M).$$

In the above, it was assumed that $\pi : E \rightarrow M$ was a real vector bundle. If we instead treat it as a complex vector bundle, we get the similarly defined Chern classes.

Definition 7.11. For an n -dimensional complex vector bundle $\pi : E \rightarrow M$, the *Chern class* of degree k is the cohomology class

$$\left[\left(\frac{-1}{2\pi i} \right)^k \sigma_k(E) \right] \in H^{2k}(M; \mathbb{R}),$$

and the *total Chern class* is

$$c(E) = 1 + c_1(E) + \cdots + c_n(E) \in H^*(M; \mathbb{R}).$$

Note that we've utilized the isomorphism $H_{DR}^k(M) \equiv H^k(M; \mathbb{R})$, and that, in fact, each Chern class is a real cohomology class. The proof of this fact parallels that of $[f(E)] = 0$ whenever f has odd degree. Before moving on to a purely homological definition, we give one of many connections between Pontrjagin and Chern class:

$$\begin{aligned} 1 - p_1 + p_2 - \cdots + (-1)^n p_n = \\ (1 + c_1 + c_2 + \cdots + c_n)(1 - c_1 + c_2 - \cdots + (-1)^n c_n) \end{aligned}$$

We note here that Pontrjagin and Chern classes can be defined in terms of one another. There are also Stiefel-Whitney classes, usually defined axiomatically, that can then be used to define Pontrjagin and Chern classes as restrictions of fibre bundles to either real or complex situations, etc. The details can be found in [H] available online. Euler classes

$e(E)$ are another sort of characteristic class for oriented manifolds. In the real case, this class is the “square root” of the top Pontrjagin class, i.e. $e(E)$ is defined by $e(E)^2 = p_n(E) \in H^{4n}(M; \mathbb{R})$.